Hierarchical Fractal Weyl Laws for Chaotic Resonance States in Open Mixed Systems

M. J. Körber,[1,2] M. Michler,[1] A. Bäcker,[1,2] and R. Ketzmerick[1,2]

[1]*Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany*
[2]*Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany*

(Dated: April 9, 2013)

In open chaotic systems the number of long-lived resonance states obeys a fractal Weyl law, which depends on the fractal dimension of the chaotic saddle. We study the generic case of a mixed phase space with regular and chaotic dynamics. We find a hierarchy of fractal Weyl laws, one for each region of the hierarchical decomposition of the chaotic phase-space component. This is based on our observation of hierarchical resonance states localizing on these regions. Numerically this is verified for the standard map and a hierarchical model system.

PACS numbers: 05.45.Mt, 03.65.Sq, 05.45.Df

It is just a century ago that Hermann Weyl published his celebrated theorem on the asymptotic distribution of eigenmodes of the Helmholtz equation in a bounded domain [1] which has found fundamental applications in the context of acoustics, optical cavities and quantum billiards [2–4]. For a quantum billiard with a d dimensional phase space the number $\mathcal{N}(k)$ of eigenmodes with a wave number below k is on average and in the limit of large k given by $\mathcal{N}(k) \sim k^{d/2}$ up to corrections of higher order [5–9]. Only recently, this has been addressed for open scattering systems, where for the case of fully chaotic systems a fractal Weyl law was found [10–20]. Due to the opening of the system one classically obtains a fractal chaotic saddle (sometimes also called repeller), which is the invariant set of points in phase space that do not escape, neither in the future nor in the past [21, 22]. Its fractal dimension δ plays an important role quantum mechanically: The number $\mathcal{N}$ of long-lived resonance states is given by a fractal Weyl law,

$$\mathcal{N}(h) \sim h^{-\delta/2}, \tag{1}$$

which here is stated for open chaotic maps, where the k dependence is replaced by the dependence on the effective size of Planck's cell h.

Generic Hamiltonian systems exhibit a mixed phase space where regular and chaotic motion coexist [23]. Regular resonance states of the open system obey a standard Weyl law, and naively one would expect that the chaotic resonance states follow the fractal Weyl law, Eq. (1). This ignores, however, that the dynamics in the chaotic region of generic two-dimensional maps is dominated by partial transport barriers, see Fig. 1. They are hierarchically organized with decreasing fluxes towards the regular regions [24–28]. They strongly impact the system's classical [24–30] and quantum mechanical [31–41] properties, and lead to, e. g., the localization of eigenstates in phase space [31–33, 37, 41] and fractal conductance fluctuations [34, 35, 39].

Classically, the chaotic saddle in generic systems gives rise to an individual fractal dimension for each region of the hierarchical decomposition of phase space [29]. These are *effective* fractal dimensions as they are scale dependent and coincide in the limit of arbitrarily small scales. Quantum mechanically, fractal Weyl laws for open systems with a mixed phase space have been investigated in Refs. [42–45], but the influence of the hierarchical phase-space structure remains open. In particular, the individual effective fractal dimensions of the chaotic saddle have not been taken into account, so far.

In this paper we propose a generalization of the Weyl law to open systems with a mixed phase space. We find *hierarchical fractal Weyl laws*,

$$\mathcal{N}_j(h) \sim h^{-\delta_j/2}, \tag{2}$$

one for each phase-space region A_j of the hierarchical decomposition of the chaotic component in a generic two-dimensional phase space. Here, δ_j denotes the effective fractal dimension of the chaotic saddle in each region. Quantum

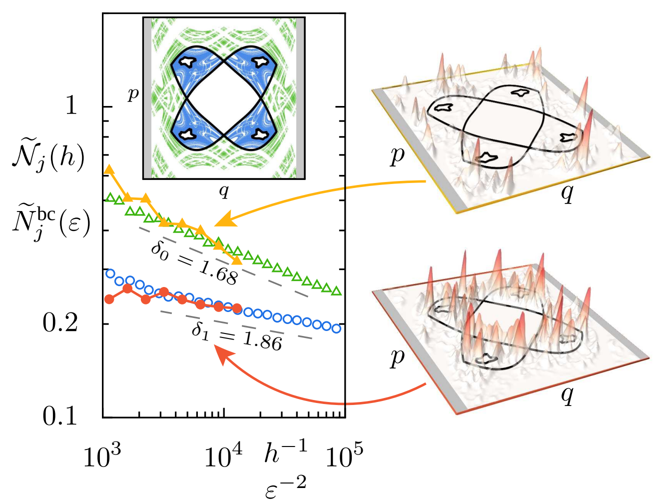

FIG. 1. (Color online) Hierarchical fractal Weyl laws $\widetilde{\mathcal{N}}_j$ vs. h^{-1} (filled symbols) counting hierarchical resonance states in the outer (A_0, triangles) and inner (A_1, circles) chaotic regions, enclosed by partial barriers (solid lines, inset), for the standard map at $\kappa = 2.9$, and corresponding typical Husimi representations (right) for $h = 1/1000$. They are compared to the box-counting scaling $\widetilde{N}_j^{\mathrm{bc}}$ vs. ε^{-2} (open symbols) giving distinct fractal dimensions δ_j of the corresponding regions of the chaotic saddle (inset). The rescaled counting functions are given by $\widetilde{\mathcal{N}}_j(h) = \mathcal{N}_j(h) \cdot h \cdot f_j$ and $\widetilde{N}_j^{\mathrm{bc}}(\varepsilon) = N_j^{\mathrm{bc}}(\varepsilon) \cdot \varepsilon^2$, respectively.

mechanically, this result is based on our observation of *hierarchical resonance states*, which predominantly localize on one of the regions A_j. Their number $\mathcal{N}_j$ follows the hierarchical fractal Weyl laws, Eq. (2). This is confirmed for the generic standard map and a hierarchical model system.

Classical properties: We first review the classical properties of the chaotic saddle in a generic mixed system and illustrate them for the prototypical example of the Chirikov standard map [46]. It is obtained from the kicked rotor Hamiltonian $H(q,p,t) = T(p)+V(q)\sum_{n\in\mathbb{Z}}\delta(t-n)$ with kinetic energy $T(p) = p^2/2$ and kick potential $V(q) = \frac{\kappa}{4\pi^2}\cos(2\pi q)$. At integer times t it leads to the symmetrized map $q_{t+1} = q_t + T'(p^*)$, $p_{t+1} = p^* - \frac{1}{2}V'(q_{t+1})$ with $p^* = p_t - \frac{1}{2}V'(q_t)$ on the torus $[0,1) \times \left[-\frac{1}{2},\frac{1}{2}\right)$. We open the system by defining absorbing stripes of width 0.05 on the left and right, see Fig. 1. This leads to a chaotic saddle Γ, for which a finite time approximation is shown in Fig. 1 for $\kappa = 2.9$. The regular islands (including small embedded chaotic layers), which appear on all scales in phase space, form an invariant set of positive measure. The chaotic saddle Γ of the open system is strongly structured by the presence of partial barriers. They originate from Cantori or stable/unstable manifolds of hyperbolic periodic orbits [28]. Partial barriers provide a hierarchical tree-like decomposition [27] of the chaotic component of phase space into regions A_j. Figure 1 shows the partial barriers surrounding the largest two regions A_0 and A_1 (which are quantum mechanically accessible.) They are all constructed from stable/unstable manifolds of a period 4 and a period 28 orbit.

Using the box-counting method [47] one can associate a fractal dimension δ_j to the intersection $\Gamma \cap A_j$ of the chaotic saddle Γ with each of the regions A_j. The number $N_j^{\mathrm{bc}}(\varepsilon)$ of occupied boxes of side length ε scales like $N_j^{\mathrm{bc}}(\varepsilon) \sim \varepsilon^{-\delta_j}$, see Fig. 1, with $\delta_0 = 1.68$ and $\delta_1 = 1.86$. To emphasize the difference between such dimensions close to two, the ordinate is rescaled by ε^2. The abscissa is chosen to be ε^{-2} which will be convenient for the comparison to the fractal Weyl law. The increase of δ_j towards two when going deeper into the hierarchy can be qualitatively understood by adapting the Kantz–Grassberger relation [48] from fully chaotic systems. The asymptotic behavior for ε approaching zero will be discussed later in connection to the semiclassical limit.

Quantum classification of states: We now present the essential quantum effect that resonance states localize predominantly on one of the regions A_j. The closed quantum system is described by the time-evolution operator $U = \exp\{-\frac{i}{2h}V(q)\}\exp\{-\frac{i}{h}T(p)\}\exp\{-\frac{i}{2h}V(q)\}$. The corresponding open quantum system is given by $U_{\mathrm{open}} = PUP$, where P is a projector on all positions not in the absorbing regions. The resonance states ψ are given by $U_{\mathrm{open}}\psi = \exp[-i(\varphi - i\gamma/2)]\psi$. Regular resonance states are predominantly located in the regular region. Chaotic resonance states are predominantly located in either of the hierarchical regions A_j, see Fig. 1. Hence, we will call them *hierarchical resonance states* (of region A_j.) Such a localization of chaotic eigenstates on different sides of a partial barrier is well known for closed quantum systems [24, 33, 38, 41]. Chaotic eigenstates localized in the hierarchical region of a mixed phase space were termed hierarchical states [37]. They require that the classical flux Φ across a partial barrier is small compared to Planck's constant h, i. e. $\Phi \ll h$, while in the opposite case eigenstates would be equidistributed ignoring the partial barrier. Quite surprisingly, we find that this condition from closed systems is irrelevant for hierarchical resonance states in open quantum systems. In the standard map at $\kappa = 2.9$ we have $\Phi \approx 1/80$, and for much smaller $h = 1/1000$ typical resonance states still predominantly localize in one of the regions A_j, as shown in Fig. 1. Even for h as small as $h = 1/12800$, see Fig. 2 (insets), this is the case. This crucial phenomenon for our study highlights the strong impact of the opening. For the transition between localization and equidistribution of resonance states in the hierarchical region of open systems we conjecture that it depends on the ratio of escape rates of neighboring regions. If the escape rates and, thus, the corresponding quantum decay rates of resonance states localized on different sides of the partial barrier are sufficiently different, such that the distance of their resonance energies exceeds the coupling strength, the hierarchical resonance states do not hybridize. In contrast, for the closed system the distance of the corresponding real eigenenergies would be much smaller and lead to hybridization for $h < \Phi$. This impact of the opening will be studied quantitatively in the future.

For the present study it is sufficient to observe that the great majority of chaotic resonance states is predominantly located in one of the regions A_j, allowing their classification. Numerically, we use their relative local Husimi weight in A_j (in the case of A_0 excluding the area of the opening) and discard states with more than 50% Husimi weight in the regular region

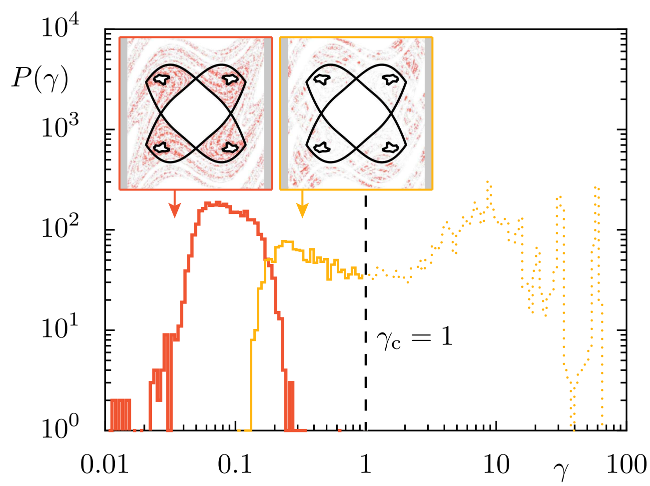

FIG. 2. (Color online) Distributions $P(\gamma)$ of decay rates γ for hierarchical resonance states of the standard map at $\kappa = 2.9$ located in regions A_0 (right) and A_1 (left) for $1/h = 12800$ and corresponding Husimi representations of typical states (insets). Short-lived states ($\gamma > \gamma_{\mathrm{c}}$) are not counted in the fractal Weyl law.

and the deep hierarchical region (A_j, $j \geq 2$). The validity of this classification is supported by the distribution of the decay rates γ of the corresponding resonance states, see Fig. 2. States which are located deeper in the hierarchy have smaller decay rates. In fact, the two distributions for regions A_0 and A_1 have a small overlap, only. Note, that an alternative classification of resonance states purely based on their decay rates γ, would fail deeper in the hierarchy, as the tree-like structure allows for different regions A_j having strongly overlapping decay rate distributions.

Hierarchical Fractal Weyl laws: For each region A_j of the hierarchical phase space we now relate the number $\mathcal{N}_j$ of hierarchical resonance states of that region to the fractal dimension δ_j of the chaotic saddle in that region. For this we use the fractal Weyl law of fully chaotic systems [11, 12], Eq. (1), individually for each region A_j. This gives our main result that in open systems with a mixed phase space one obtains a hierarchy of fractal Weyl laws, one for each phase-space region A_j, Eq. (2). We stress that this result is based on the surprising existence of hierarchical resonance states. Note, that as a consequence of Eq. (2) the total number of long-lived hierarchical resonance states is a superposition of power laws with different exponents and not a single power law. We will discuss the range of validity of the hierarchical Weyl laws and the semiclassical limit further below.

To give an intuitive understanding of the hierarchical fractal Weyl laws, let us recall the interpretation of the fractal Weyl law [11], and apply it in the presence of a hierarchical phase space. The number of quantum states localizing on a particular phase-space region is given by the number of Planck cells necessary to cover the chaotic saddle in that region. Using the scaling, $N_j^{\mathrm{bc}}(\varepsilon) \sim \varepsilon^{-\delta_j}$, of the number of boxes N_j^{bc} to cover the chaotic saddle in region A_j and the identification of the box area ε^2 with the Planck cell area h directly leads to Eq. (2). In analogy to fully chaotic open systems [49], the fine-scale structure of the hierarchical resonance states is dominated by the backward trapped set for right eigenstates of U_{open} and the forward trapped set for left eigenstates, see insets in Fig. 2.

The numerical investigation of the standard map supports the existence of hierarchical fractal Weyl laws, as we now show. By the classification of resonance states we are able to determine the number $\mathcal{N}_j(h)$ of long-lived hierarchical resonance states associated to a particular region A_j depending on h. We restrict ourselves to the consideration of small h such that $\Phi/h \gtrsim 10$ where quantum mechanics can very well mimic classical transport in phase space [41]. Short-lived states are discarded by defining an arbitrary cut-off rate $\gamma_{\mathrm{c}} = 1$, as usual for the fractal Weyl law [12]. Here this affects resonance states of the outermost region A_0 only. We obtain distinct behavior for each counting function $\mathcal{N}_j$, see Fig. 1. Here the ordinate is rescaled by h and the abscissa is chosen to be h^{-1}, corresponding to the previous classical rescalings. For small h one observes the power-law scaling of Eq. (2) and good agreement with the box-counting results for the fractal dimensions δ_j of $\Gamma \cap A_j$. Note that we fitted prefactors f_j to the quantum results to better demonstrate that they scale with power-law exponents in agreement with the classical counterparts. Both prefactors f_j are of order one ($f_0 = 2.6$, $f_1 = 0.85$). Figure 1 confirms for two regions A_j of the standard map that they give rise to hierarchical fractal Weyl laws.

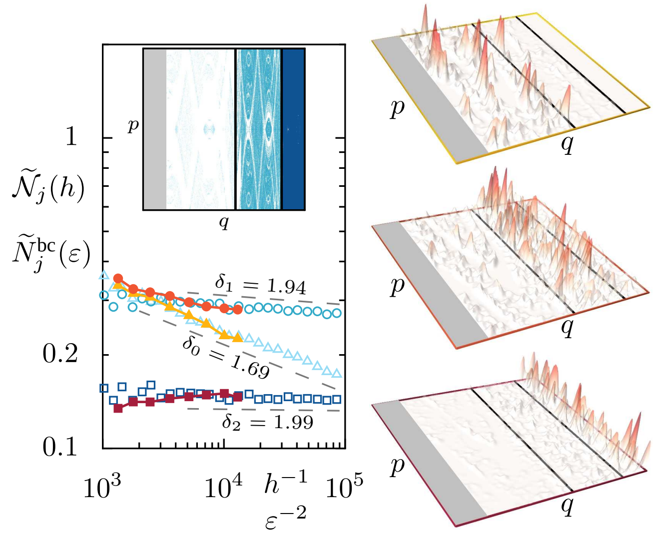

FIG. 3. (Color online) Hierarchical fractal Weyl laws $\widetilde{\mathcal{N}}_j$ vs. h^{-1} (filled symbols) counting hierarchical resonance states in the outer (A_0, triangles), central (A_1, circles), and inner (A_2, squares) chaotic regions for the hierarchical model system, and corresponding typical Husimi representations (right) for $h = 1/1115$. They are compared to the box-counting scaling $\widetilde{N}_j^{\mathrm{bc}}$ vs. ε^{-2} (open symbols) giving distinct fractal dimensions δ_j of the corresponding regions of the chaotic saddle (inset). Here we used $f_0 = 1.75$, $f_1 = 1.55$, and $f_2 = 0.8$ which again are of order one.

Hierarchical model system: To verify the hierarchical fractal Weyl laws for more than two regions, we suggest the following system that models the hierarchical structure of partial barriers in a generic mixed phase space, similar in spirit to a one-dimensional model [29] and a Markov chain [26]. In contrast to the standard map, this designed model quantum mechanically allows for studying three regions numerically.

We first define a composed symplectic map $C \circ M$ on the phase space $\Gamma = [0, 1) \times [0, 1)$. It models b partial barriers at the positions $q_1 < \cdots < q_b$ as straight lines in p-direction, giving a decomposition of Γ into $b + 1$ regions $A_j = [q_j, q_{j+1}) \times [0, 1)$ with $q_0 = 0$ and $q_{b+1} = 1$. The map M describes the uncoupled dynamics being sufficiently mixing in each region A_j. We choose the standard map at kicking strength $\kappa = 10$ acting individually on each of the regions A_j after appropriate rescaling. The map C couples these regions mimicking the turnstile mechanism of a partial barrier with flux Φ_j by exchanging the areas $[q_j - \Phi_j, q_j) \times [0, 1)$ with their neighboring areas $[q_j, q_j + \Phi_j) \times [0, 1)$. Finally, we open the system by defining the absorbing region $[0, \Phi_0) \times [0, 1)$ and the corresponding escape map O, leading to the hierarchical

model map $O \circ C \circ M \circ O$. The barrier positions q_j are chosen such that the area $\mu(A_j) = q_{j+1} - q_j$ of the regions A_j obeys the scaling $\mu(A_j)/\mu(A_{j+1}) = \alpha$ with some $\alpha \geq 1$. We choose the scaling behavior of the fluxes as $\Phi_j/\Phi_{j+1} = \phi$ with some $\phi \geq \alpha$, such that even for an arbitrary deep hierarchy, the size of the coupling regions will not exceed the size of the regions A_j. In this paper we use the parameters $b = 2$, $\alpha = 2$, $\phi = 4$, and $\Phi_0/\mu(A_0) = 1/4$. Hence, we have the fluxes $\Phi_1 = 1/28$ and $\Phi_2 = 1/112$.

Figure 3 shows the results for the hierarchical model system: We obtain the fractal dimensions $\delta_0 = 1.69$, $\delta_1 = 1.94$, and $\delta_2 = 1.99$. Quantum mechanically, we again find hierarchical resonance states predominantly localizing on one of the regions A_j, even though $h \ll \Phi_1, \Phi_2$. There number follows the proposed hierarchical fractal Weyl laws according to Eq. (2).

Semiclassical limit: We now discuss the range of validity of the individual fractal Weyl law, Eq. (2), for some region A_j as a function of decreasing Planck's constant h. Once h is small enough to resolve this region A_j, i.e. $h \ll A_j$, there are four relevant regimes: (i) For h larger than the greatest flux Φ_j across its surrounding partial barriers, $h > \Phi_j$, one has resonance states localized on region A_j with just a small coupling to other regions, as for closed systems [33, 41]. Consequently, the number of resonance states in this regime scales with the usual Weyl law as h^{-1}. The resonance states do not resolve the fine-scale structure of the chaotic saddle. (ii) For h smaller than the flux, $h < \Phi_j$, the resonance states still localize in region A_j. Furthermore, they begin to resolve the fractal structure of the chaotic saddle. This is the main regime discussed in this paper and described by Eq. (2) with a fractal dimension δ_j of the intersection of the chaotic saddle with region A_j. (iii) For much smaller values of h, we expect that hierarchical resonance states start to extend over several regions A_j. This expectation is based on the fact that the effective fractal dimensions of these regions coincide for sufficiently small side length ε of the boxes, due to their coupling across partial barriers with the largest dimension dominating [29]. As a consequence we expect that the quantum decay rates become identical leading to a hybridization of the resonance states. This classical scale ε implies the beginning of this regime on the quantum scale $h = \varepsilon^2$. Owing to an extremely slow ε dependence of the effective fractal dimensions these quantum mechanical consequences are beyond current numerical investigations. (iv) In the semiclassical limit, $h \to 0$, we presume an overall Weyl law for the hierarchical region with the number of resonance states scaling as h^{-1}. This is based on the fact that asymptotically ($\varepsilon \to 0$), the fractal dimension is the same in all regions A_j [29], and that for an infinite hierarchical structure of partial barriers, the fractal dimension of the chaotic saddle equals two [50].

Discussion: In this paper we have shown that the hierarchical fractal Weyl laws, Eq. (2), describe the important regime (ii) where hierarchical resonance states predominantly localize on one of the regions A_j and resolve the fractal structure of the chaotic saddle. Note that Eq. (2) also applies to the other regimes by choosing the phase-space regions according to the predominant localization of resonance states.

There is also a related Weyl law with fractional exponent in the presence of a hierarchical phase space even for closed systems. In Ref. [37] it was shown that the number of hierarchical states scales as $h^{1-1/\gamma}$, relating this Weyl law to the exponent γ of the power-law decay of the Poincaré recurrence-time distribution $P(t) \sim t^{-\gamma}$ with $\gamma > 1$. These hierarchical states localize on regions A_j with flux $\Phi_j < h$. In the open system they correspond to resonance states of all regions in regime (i), which, thus, fulfill the same fractional Weyl law $\sim h^{1-1/\gamma}$. The idea of connecting the fractal Weyl law with the power-law decay of $P(t)$ was also studied in Ref. [44]. There a fractal Weyl law $\sim h^{\gamma-1}$ was found for the number of resonance states fulfilling the cut-off criterion $\gamma < \gamma_c \sim h$. It can be shown that this includes resonance states from regimes (i) and (ii) when applied to a hierarchical phase-space structure. In contrast, the hierarchical fractal Weyl laws, Eq. (2), focus on resonance states of regime (ii). There, even though $h < \Phi_j$, the resonance states predominantly localize on one region A_j. Furthermore, these resonance states resolve the fine-scale structure of the chaotic saddle. A future challenge is the study of fractal Weyl laws in higher dimensional systems with a generic phase space.

We are grateful to E. G. Altmann, H. Kantz, H. Schomerus, A. Shudo, and T. Tél for helpful comments and stimulating discussions, and acknowledge financial support through the DFG Forschergruppe 760 "Scattering systems with complex dynamics."

[1] H. Weyl, Math. Ann. **71**, 441 (1912).
[2] M. Kac, Amer. Math. Mon. **73**, 1 (1966).
[3] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, 1999).
[4] W. Arendt et al., in *Mathematical Analysis of Evolution, Information, and Complexity*, edited by W. Arendt and W.-P. Schleich (Wiley-VCH, Weinheim, 2009) pp. 1–71.
[5] R. Balian and C. Bloch, Ann. Phys. **60**, 401 (1970).
[6] H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (B.-I. Wissenschaftsverlag, Mannheim, 1976).
[7] M. V. Berry and C. J. Howls, Proc. R. Soc. Lond. A **447**, 527 (1994).
[8] V. Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer Monographs in Mathematics (Springer, Berlin, 2010).
[9] A. Bäcker et al., Euro. Phys. Lett. **94**, 30004 (2011).
[10] K. K. Lin and M. Zworski, Chem. Phys. Lett. **355**, 201 (2002).
[11] W. T. Lu, S. Sridhar, and M. Zworski, Phys. Rev. Lett. **91**, 154101 (2003).
[12] H. Schomerus and J. Tworzydło, Phys. Rev. Lett. **93**, 154102 (2004).
[13] S. Nonnenmacher and M. Zworski, J. Phys. A **38**, 10683 (2005).
[14] S. Nonnenmacher and M. Rubin, Nonlinearity **20**, 1387 (2007).
[15] J. Wiersig and J. Main, Phys. Rev. E **77**, 036205 (2008).
[16] D. L. Shepelyansky, Phys. Rev. E **77**, 015202(R) (2008).
[17] J. A. Ramilowski, S. D. Prado, F. Borondo, and D. Farrelly, Phys. Rev. E **80**, 055201(R) (2009).

[18] A. Eberspächer, J. Main, and G. Wunner, Phys. Rev. E **82**, 046201 (2010).
[19] L. Ermann and D. L. Shepelyansky, Eur. Phys. J. B **75**, 299 (2010).
[20] A. Potzuweit et al., Phys. Rev. E **86**, 066205 (2012).
[21] Y.-C. Lai and T. Tél, *Transient Chaos: Complex Dynamics on Finite-Time Scales*, Applied Mathematical Sciences (Springer, New York, 2011).
[22] E. G. Altmann, J. S. E. Portela, and T. Tél, Rev. Mod. Phys. in press (2013) arXiv:1208.0254.
[23] L. Markus and K. R. Meyer, *Generic Hamiltonian Dynamical Systems are neither Integrable nor Ergodic*, Memoirs of the Mathematical Society, Vol. 144 (Amer. Math. Soc., Providence, Rhode Island, 1974).
[24] R. S. MacKay, J. D. Meiss, and I. C. Percival, Phys. Rev. Lett. **52**, 697 (1984).
[25] R. S. MacKay, J. D. Meiss, and I. C. Percival, Physica **13D**, 55 (1984).
[26] J. D. Hanson, J. R. Cary, and J. D. Meiss, J. Stat. Phys. **39**, 327 (1985).
[27] J. D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985).
[28] J. D. Meiss, Rev. Mod. Phys. **64**, 795 (1992).
[29] A. E. Motter, A. P. S. de Moura, C. Grebogi, and H. Kantz, Phys. Rev. E **71**, 036215 (2005).
[30] G. Cristadoro and R. Ketzmerick, Phys. Rev. Lett. **100**, 184101 (2008).
[31] R. C. Brown and R. E. Wyatt, Phys. Rev. Lett. **57**, 1 (1986).
[32] T. Geisel, G. Radons, and J. Rubner, Phys. Rev. Lett. **57**, 2883 (1986).
[33] O. Bohigas, S. Tomsovic, and D. Ullmo, Phys. Rep. **233**, 43 (1993).
[34] R. Ketzmerick, Phys. Rev. B **54**, 10841 (1996).
[35] A. S. Sachrajda et al., Phys. Rev. Lett. **80**, 1948 (1998).
[36] N. T. Maitra and E. J. Heller, Phys. Rev. E **61**, 3620 (2000).
[37] R. Ketzmerick, L. Hufnagel, F. Steinbach, and M. Weiss, Phys. Rev. Lett. **85**, 1214 (2000).
[38] A. Bäcker, A. Manze, B. Huckestein, and R. Ketzmerick, Phys. Rev. E **66**, 016211 (2002).
[39] A. P. Micolich et al., Phys. Rev. B **70**, 085302 (2004).
[40] C.-M. Goletz, F. Grossmann, and S. Tomsovic, Phys. Rev. E **80**, 031101 (2009).
[41] M. Michler, A. Bäcker, R. Ketzmerick, H.-J. Stöckmann, and S. Tomsovic, Phys. Rev. Lett. **109**, 234101 (2012).
[42] M. Kopp and H. Schomerus, Phys. Rev. E **81**, 026208 (2010).
[43] M. E. Spina, I. García-Mata , and M. Saraceno, J. Phys. A **43**, 392003 (2010).
[44] A. Ishii, A. Akaishi, A. Shudo, and H. Schomerus, Phys. Rev. E **85**, 046203 (2012).
[45] C. Birchall and H. Schomerus, arXiv:1208.2259 (2012).
[46] G. Casati et al., in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, Lecture Notes in Physics, Vol. 93 (Springer, Berlin, 1979) pp. 334–352.
[47] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed. (Wiley, West Sussex, 2003).
[48] H. Kantz and P. Grassberger, Physica **17D**, 75 (1985).
[49] J. P. Keating, M. Novaes, S. D. Prado, and M. Sieber, Phys. Rev. Lett. **97**, 150406 (2006).
[50] Y.-T. Lau, J. M. Finn, and E. Ott, Phys. Rev. Lett. **66**, 978 (1991).